Solution Sheet 5

Exercise 1. In this exercise we will show that the Riemann surface $\mathbb{CP}^1$ is a $SU(2)$-homogeneous space.

1. Show that the action of $SU(2)$ on $\mathbb{CP}^1$ defined by $g \cdot [v] = [gv]$ is transitive.
2. Show that
$$\mathrm{Stab}_{SU(2)}([1:0]) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong S^1,$$
and deduce that
$$\Psi : SU(2)/S^1 \to \mathbb{CP}^1, \quad gS^1 \mapsto g \cdot [1:0]$$
is well-defined and a bijection.
3. Pull back the complex structure of $\mathbb{CP}^1$ along Ψ to $SU(2)/S^1$ and conclude that Ψ is a biholomorphism.

Solution 1.

1. If $[v] = [w]$, then $w = \lambda v$ for some $\lambda \in \mathbb{C}^\times$, so $gw = \lambda gv$ implies $[gw] = [gv]$, so the action respects projective equivalence. Given $[v], [w] \in \mathbb{CP}^1$, let $v, w \in \mathbb{C}^2$ be representatives with $\|v\| = \|w\| = 1$. We can extend v to an orthonormal basis $\{v, v'\}$ and similarly w to $\{w, w'\}$. Then we define a linear map g by
$$g(v) = w, \quad g(v') = w'.$$
So g maps an orthonormal basis to an orthonormal basis, hence g is unitary, that is, $g \in U(2)$. To ensure $g \in SU(2)$, we can scale. Namely, if $\det g = e^{i\theta}$, then we set $g' = e^{-i\theta/2} g \in SU(2)$. Then we have $g'v = e^{-i\theta/2} w$, which implies $[g'v] = [w]$, so the action is transitive.
2. We can write an element of $SU(2)$ as
$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1.$$
This implies that $g \cdot [1:0] = [a : -\bar{b}]$. This equals $[1:0]$ if and only if $b = 0$, which then forces $|a| = 1$. Thus
$$\mathrm{Stab}_{SU(2)}([1:0]) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong S^1.$$
We now show that the map Ψ is well-defined and bijective.
   - *Well defined.* If $g' = gh$ with $h \in S^1 = \mathrm{Stab}_{SU(2)}([1:0])$, then
   $$\Psi(g'S^1) = g' \cdot [1:0] = g \cdot (h \cdot [1:0]) = g \cdot [1:0] = \Psi(gS^1).$$
   - *Surjective.* We showed that the $SU(2)$-action on $\mathbb{CP}^1$ is transitive, so every point is $g \cdot [1:0]$ for some $g \in SU(2)$.
   - *Injective.* If $\Psi(g_1 S^1) = \Psi(g_2 S^1)$, then $g_2^{-1} g_1$ stabilizes $[1:0]$, hence $g_2^{-1} g_1 \in S^1$. Thus $g_1 S^1 = g_2 S^1$.
3. We consider the standard charts (U_0, z_0) and (U_∞, z_∞) on $\mathbb{CP}^1$ defined by
$$z_0([z_0 : z_1]) = \frac{z_1}{z_0} \text{ on } U_0 = \{[z_0 : z_1] \in \mathbb{CP}^1 : z_0 \neq 0\},$$
$$z_\infty([z_0 : z_1]) = \frac{z_0}{z_1} \text{ on } U_\infty = \{[z_0 : z_1] \in \mathbb{CP}^1 : \ z_1 \neq 0\}.$$

On the overlap $U_0 \cap U_\infty$ we have $z_\infty = \frac{1}{z_0}$, which is holomorphic on $\mathbb{C} \setminus \{0\}$. Via Ψ we pull back these charts to $SU(2)/S^1$:

$$\widetilde{U}_i := \Psi^{-1}(U_i), \qquad \widetilde{z}_i := z_i \circ \Psi : \widetilde{U}_i \to \mathbb{C},$$

with $i \in \{0, \infty\}$. Then the transition maps on overlaps are

$$\widetilde{z}_\beta \circ \widetilde{z}_\alpha^{-1} = (z_\beta \circ \Psi) \circ (z_\alpha \circ \Psi)^{-1} = z_\beta \circ z_\alpha^{-1},$$

which are holomorphic. Thus these charts endow $SU(2)/S^1$ with a complex structure, and Ψ is biholomorphic by construction.

Exercise 2. Let $C_P \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth projective curve defined by a homogeneous polynomial $F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$. Consider the standard affine charts $U_Z = \{Z \neq 0\}$ and $U_Y = \{Y \neq 0\}$. Suppose $[x_0 : y_0 : 1] \in C_P \cap U_Z$ is a smooth point with $\partial P_y(x_0, y_0) \neq 0$, where $P(x, y) = F(x, y, 1)$.

(1) Show that the map $x \mapsto [x : y(x) : 1]$ defines a local holomorphic parametrization of C_P near $[x_0 : y_0 : 1]$.

(2) Express the parametrization from part (1) in coordinates of the affine chart U_Y.

Solution 2. (1) Since $\partial_y P(x_0, y_0) \neq 0$, by the analytic implicit function theorem, there exists a neighborhood U of x_0, a holomorphic function $y(x)$ defined on U such that $P(x, y(x)) = 0$ and $y(x_0) = y_0$ for all $x \in U$. Therefore the map

$$x \mapsto [x : y(x) : 1]$$

defines a local holomorphic parametrization of the curve C_P near the point $[x_0 : y_0 : 1]$.

(2) In the chart U_Y, a point $[X : Y : Z] \in \mathbb{P}^2$ with $Y \neq 0$ has coordinates $u = X/Y$, $v = Z/Y$. Because $[x : y(x) : 1] = [x/y(x) : 1 : 1/y(x)]$, the parametrization becomes $x \mapsto \left(\frac{x}{y(x)}, \frac{1}{y(x)}\right)$.

Exercise 3.
Let $U_X = \{X \neq 0\}$, $U_Y = \{Y \neq 0\}$, $U_X = \{Z \neq 0\}$ and identify $\mathbb{C}^2$ with U_2 via $(x, y) = (X/Z, Y/Z)$. For $P(x, y) \in \mathbb{C}[x, y]$, let $\overline{C_P} \subset \mathbb{P}^2_{\mathbb{C}}$ be the Z-homogenized projective closure.

(1) Let $P(x, y) = x^2 + y^2 - 1$. Find the homogeneous equation of $\overline{C_P}$ and show that $\overline{C_P} \cap U_i$ is biholomorphic to $\mathbb{C} \backslash \{0\}$ for $i = X, Y, Z$.

(2) Let $P(x, y) = y^2 - \prod_{j=1}^n (x - x_j)$ with distinct $x_j \in \mathbb{C}$ and $n > 2$. Find the homogeneous equation of $\overline{C_P}$ and determine $\overline{C_P} \backslash C_P$.

Solution 3.

(1) The homogeneous equation of $\overline{C_P}$ is $X^2 + Y^2 - Z^2 = 0$. In the affine chart U_Z, we use coordinates $(x, y) = (X/Z, Y/Z)$, so that the curve equations becomes $x^2 + y^2 = 1$. Under the change of coordinates $u = x + iy$ and $v = x - iy$, this curve equation becomes $uv = 1$. The curve $C = \{(u, v) \in \mathbb{C}^2 \mid uv = 1\}$ is biholomorphic to $\mathbb{C} \backslash \{0\}$. Indeed, define the map

$$\varphi : \mathbb{C} \setminus \{0\} \to C, \quad \varphi(z) = (z, 1/z)$$

which is holomorphic on $\mathbb{C} \setminus \{0\}$, and whose inverse

$$\varphi^{-1} : C \to \mathbb{C} \setminus \{0\}, \quad \varphi^{-1}(u, v) = u$$

is also holomorphic. One can use a similar argument in U_X and U_Y.

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(2) We homogenizing via $x = X/Z$, $y = Y/Z$. The homogeneous equation of $\overline{C_P} \subset \mathbb{P}^2$ is then

$$Y^2 Z^{n-2} = \prod_{j=1}^{n} (X - x_j Z).$$

Setting $Z = 0$ yields $0 = X^n$, hence $X = 0$, and the only point at infinity is $[0 : 1 : 0]$. Therefore,

$$\overline{C_P} \setminus C_P = \{[0 : 1 : 0]\}.$$

Exercise 4. (for credit, due on 19 October) *(5 points)* Let $d \geq 2$. Let C_P denote the Fermat curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous polynomial $X^d + Y^d + Z^d = 0$. Let

$$f : C_P \to \mathbb{P}^1_{\mathbb{C}}, \quad [X : Y : Z] \mapsto [X : Y].$$

Compute the branch values of f.

Solution 4. Let us first look at fibers over points $[\lambda : 1] \in \mathbb{P}^1_{\mathbb{C}}$; we work in the affine chart $\{Y \neq 0\}$ (put $y = 1$). For fixed $\lambda \in \mathbb{C}$, the fiber of f over $[\lambda : 1]$ consists of points $[\lambda : 1 : z]$ with $z^d = -(\lambda^d + 1)$. For generic λ we have $\lambda^d + 1 \neq 0$, and $z^d = -(\lambda^d + 1)$ has d distinct solutions. So the fiber consists of d distinct points, and f is unramified at each of them. Ramification happens precisely when the fiber consists of less than d points, that is, when $\lambda^d + 1 = 0$. In this case, $z = 0$ and the fiber consists of the single point $[\lambda : 1 : 0]$, with ramification multiplicity d. Next we examine the fiber over $[1 : 0] \in \mathbb{P}^1_{\mathbb{C}}$. We switch to the chart $\{X \neq 0\}$. With the coordinates $y = Y/X$ and $z = Z/X$, the curve equation in this chart is $1 + y^d + z^d = 0$ and $f([1 : y : z]) = [1 : y]$. At points with $y = 0$ we have $z = \eta$ with $\eta^d = -1$ and $\partial_z(1 + y^d + z^d) = d\eta^{d-1} \neq 0$, so we can solve for $z = z(y)$ locally. Therefore f looks locally like the identity map has there is no ramification. Therefore the only branch values are $[\lambda : 1]$ with $\lambda^d = -1$.